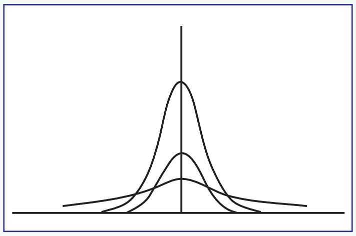
MATHEMATICAL DISCUSSION

Several devices in natural systems perform their functions by responding to an effect f by a signal $\tilde{f}$. Each device or system of this type can be represented by an operator A that transforms an input signal f into an output signal $\tilde{f}$, which is equal to Af. Naturally, each operator has its own area of perceived signals (the operator domain) and its own form of response (the codomain). A convenient mathematical model for a large class of real processes is a linear shift operator A, which is translation-invariant.

A is said to be invariant under translations (or translation invariant) if, for any function f from the domain of operator A, there exists an equality



SIDEBAR FIGURE 2: SMOOTH FUNCTIONS TO SIMULATE IMPULSE

$$A(T_{t0}f) = T(Af), \text{ while}$$

$$(Tf) \times (t) = f(t - t_0)$$

If t is time, the ratio $AT=TA$ can be interpreted as suggesting that the properties of device A are constant over time; the device response to the $f(t)$ and $f(t–t)$ signals would differ by a temporal shift only.

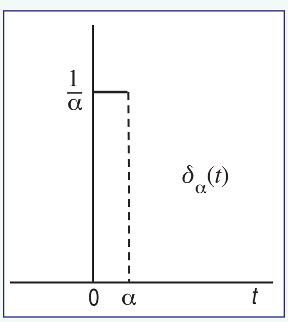
There are essentially two main objectives in using device A:

1. To anticipate the device response $\tilde{f}$ to an arbitrary input process f, *and*

2. To determine the input signal f entering the device using an output $\tilde{f}$ signal.

Let us consider the solution of the former problem using the invariant linear translation operator A. To describe the response of device A to any input f, it is sufficient to know the response E of device A to an impulse input δ.

The $E(t)$ response to a single impulse input δ is called a *device instrument function* (or a slit function in optics or unit-impulse response function in electrical engineering). Generally, the function E may be a generalized function. It is defined as a function that renders the δ function under the action of the operator A, and it can be called a fundamental solution, or Green's function, or the influence function, or the instrument function of the operator A.



SIDEBAR FIGURE 1: DISCRETE IMPULSE. This simulation becomes more accurate as the impulse's duration changes over time.

The discrete impulse can be represented, for instance, by a function shown in sidebar Figure 1, and this simulation becomes more accurate as the impulse's duration α changes over time with its total energy $\alpha \frac{1}{\alpha} = 1$ being preserved.

Instead of step functions, you can use smooth functions to simulate the impulse (sidebar Figure 2), providing certain natural conditions are met:

$$f \geq 0 \qquad \int_R f(t)dt = 1 \qquad \int_U f(t)dt \to 0$$

$$\text{with } \alpha \to 0$$

where U is a random neighborhood of t=0.

The device A response to an idealized unique impulse δ should be regarded as a function $E(t)$. The device A responses are approaching $E(t)$ as the simulation is improving δ. Naturally, this implies a certain continuity of the operator A — that is, the continuity of change in the device response $\tilde{f}$ with a continuous change of the input f.

For example, by taking the sequence $\Delta_n(t)$ of step functions $\Delta_n(t) = \delta_n(t)$ (sidebar Figure 3), and assuming that $A\Delta_n(t) = E_n$, we should obtain:

$$A\delta = E = \lim_{n\to\infty} E_n = \lim_{n\to\infty} A\Delta_n$$

Now, let us consider the input signal f (sidebar Figure 3) and the piecewise constant function shown in the same figure.

Since $l_h \to 0$ as $h \to 0$, we may assume that

$$\tilde{l}_h = Al_h \to Af = \tilde{f} \quad \text{as} \quad h \to 0$$

But if the operator A is linear and invariant to translations, then

$$\tilde{l}_h(t) = \Sigma f(\tau_i)\, E_h(t - \tau_i)\, h$$

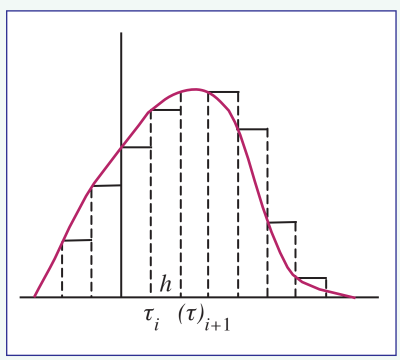
where $E_h = A\delta_n$.

Thus, as $h \to 0$, we should finally obtain

$$\tilde{f}(t) = \int_R f(\tau)E(t-\tau)d\tau \qquad (1)$$

Equation (1) solves the first of the two problems mentioned. It represents the device A response $\tilde{f}(t)$ in the form of a special integral that depends on the parameter ***t***. This integral function is fully determined by the input signal $f(t)$ and the instrument function $E(t)$ of device A.

From a mathematical viewpoint, device A and the integral (1) are the same thing. The problem of determining the input signal using the output is now reduced to a solution of the integral equation (1). This is known as the Fredholm integral equation of the first kind.



SIDEBAR FIGURE 3: THE CONTINUITY OF CHANGE